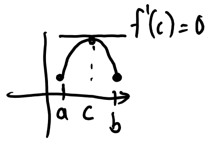
Lecture 14

Wednesday, October 5, 2016 8:15 AM

Rolle's Theorem:

Let f be a function that satisfies the following three hypotheses:

1) f is continuous on the closed interval [a,b]. ✓
2) f is differentiable on the open interval (a,b). ✓
3) f(a) = f(b). ✓



Then there is a number c in (a,b) such that f'(c) =0.

Proof of Theorem can be found in the book, Page 287

Example: Show that the function f(x) = x^3 + x - 1 has ***exactly one root***.

Step 1:
Last class, we used Intermediate Value theorem to show that the function f(x) has at least one root.

Step 2 We would like to show it has atmost 1 root.

Idea Prove it by contradiction.

Suppose f has more than 1 root.

Let a, b be roots of f i.e. $f(a)=0=f(b)$ ✓

$f(x) = x^3+x-1$ is a polynomial and therefore cont on $[a,b]$ and diff on (a,b).

Then by Rolle's Thm, there is a number c betn a & b such that $f'(c)=0$

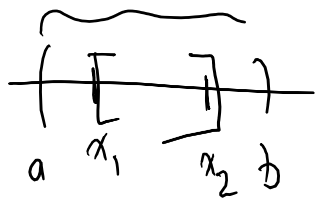
$f'(x) = 3x^2+1$, then $f'(c) = 3c^2+1$

$3c^2+1=0 \Rightarrow c^2 = -\frac{1}{3}$ (false) ✓ contradiction.

Therefore, by contradiction, f(x) has atmost 1 root.

$f'(x) > 0$

(a,b)



(x_1, x_2)

cont on (a,b)

THE MEAN VALUE THM

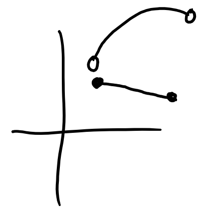
Let f be a function that satisfies the following two hypothesis(e):

1. f is cont on $[a,b]$ 2. f is diff on (a,b).

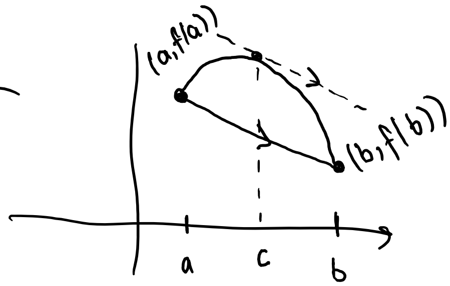
Then, there is a number c betn a & b s.t

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

secant lines



Geometrically



MVT says there is a pt c in (a,b) where tangent line to the curve at c is parallel to the secant line.

Rmk If $f(a) = f(b)$, then we get the same Statement as the Rolle's Theorem.

Rmk The proof of MVT

Basic Idea Apply Rolle's Thm to $h(x)$ where,

$$h(x) = f(x) - \frac{f(b) - f(a)}{b-a}(x-a)$$

$h(a) = 0 = h(b)$

Rolle's Thm, there is a number c in (a,b) s.t $h'(c) = 0$

Ex Spse $f(1) = 3$, and $f'(x) \leq 11$ for all values of x. How large can $f(4)$ be ?

Sdn We are given that the func f is diff everywhere, and hence continuous everywhere.

We can apply MVT on the interval $[1,4]$.

Then there is a number c betn 1 and 4 s.t

$$f'(c) = \frac{f(4)-f(1)}{4-1} \Rightarrow f'(c) = \frac{f(4)-3}{3}$$

$$\Rightarrow 3f'(c) = f(4)-3 \Rightarrow f(4) = \underline{3f'(c)+3}$$

$f'(x) \le 11$. In particular $f'(c) \le 11$.

$$f(4) = 3\underline{f'}(c)+3 \;\underline{\le}\; 3\cdot\underline{11}+3 = 36$$

- <u>Thm</u> If $f'(x) = 0$ for all x in the interval (a,b), then f is constant on (a,b). ←

Proof in book, <u>Ch 4.2</u> DIY.

<u>Cor</u> If $f'(x) = g'(x)$ for x in an interval (a,b), then $f-g$ is constant on (a,b) i.e $f(x) = g(x) + c$ (c is a constant).

<u>Pf</u> Let $H(x) = f(x) - g(x)$

$$H'(x) = f'(x) - g'(x) = 0 \text{ for all } x \text{ in } (a,b)$$

$H(x)$ is constant on (a,b) i.e $f-g$ is constant.

<u>Ch 4.3</u>

<u>Q</u> What does f' tell us about f ?

<u>Increasing / Decreasing Test</u>.

a) If $\underline{f'(x) > 0}$ on an interval, then f is increasing on that interval.

b) If $f'(x) < 0$ " " ", then " " decreasing " " ".

<u>Pf of a)</u> Pick any x_1, x_2 in that interval w/ $\underline{x_1 < x_2}$. $\Rightarrow$ $0 < x_2 - x_1$

To show function is increasing, we need to show $\underline{f(x_1) < f(x_2)}$

So f is diff on (x_1, x_2)

f is cont on $[x_1, x_2]$

So by MVT, there is a number c betn x_1 & x_2 s.t

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \Rightarrow f(x_2) - f(x_1) = \underbrace{f'(c)}_{\substack{\vee \\ 0}} \underbrace{(x_2 - x_1)}_{\substack{\vee \\ 0}}$$

$\Rightarrow f(x_2) - f(x_1) > 0 \Rightarrow f(x_2) > f(x_1)$ as desired.